

# **Transverse momentum dependent quark densities from Lattice QCD**

Bernhard Musch (Jefferson Lab)

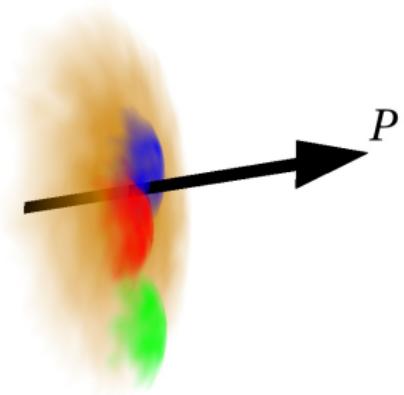
in collaboration with

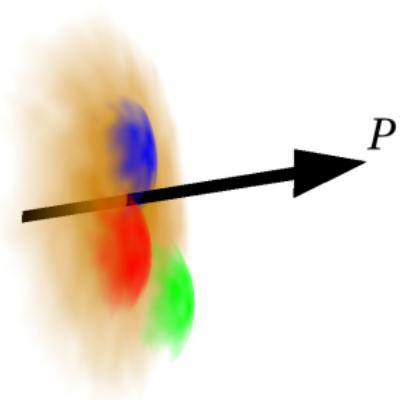
Philipp Hägler (TU München), John Negele (MIT),  
Andreas Schäfer (Univ. Regensburg),  
and the LHP Collaboration

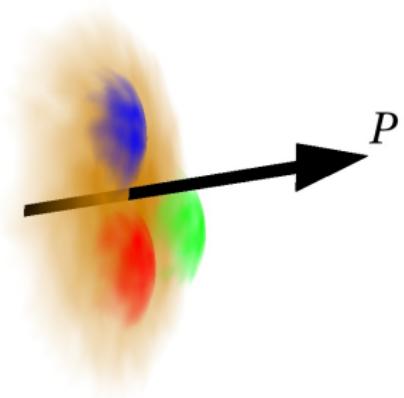
[HÄGLER ET AL. EPL88 61001 (2009)]

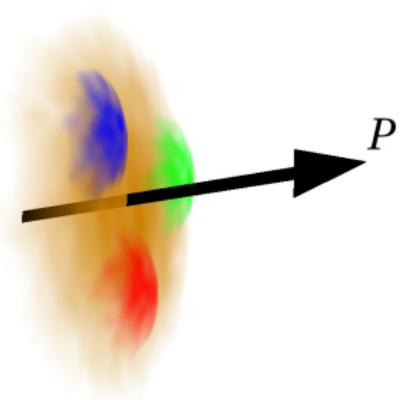
[MUSCH arXiv:0907.2381]

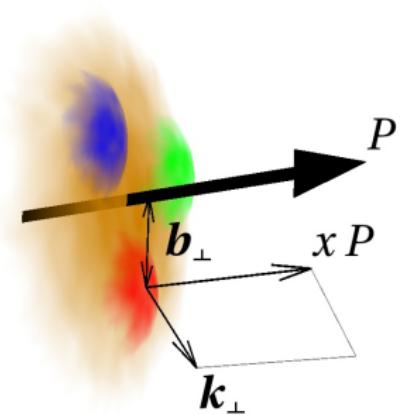


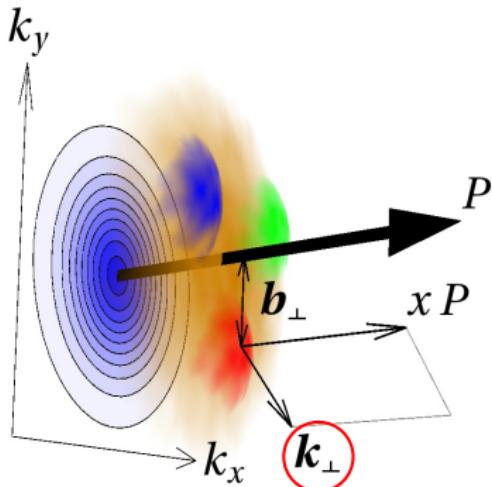












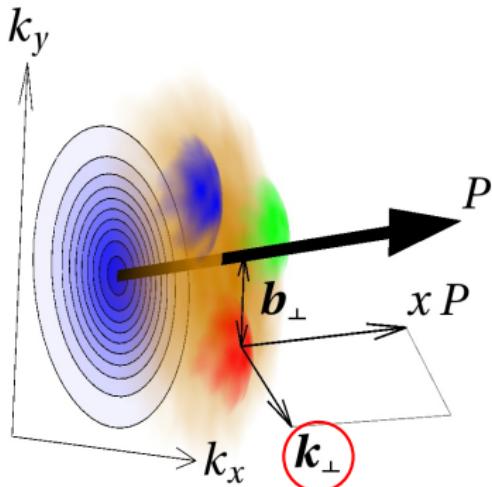
## TMD PDFs

transverse momentum dependent  
parton distribution functions

e.g.,  $f_1(x, \mathbf{k}_\perp^2)$

⇒ quark density  $\rho(\mathbf{k}_\perp)$ .

- $x$  (longitudinal momentum fraction) ⇒ PDFs
- $x, \mathbf{b}_\perp$  (impact parameter) ⇒ GPDs
- $x, \mathbf{k}_\perp$  (intrinsic transverse momentum) ⇒ TMD PDFs



## TMD PDFs

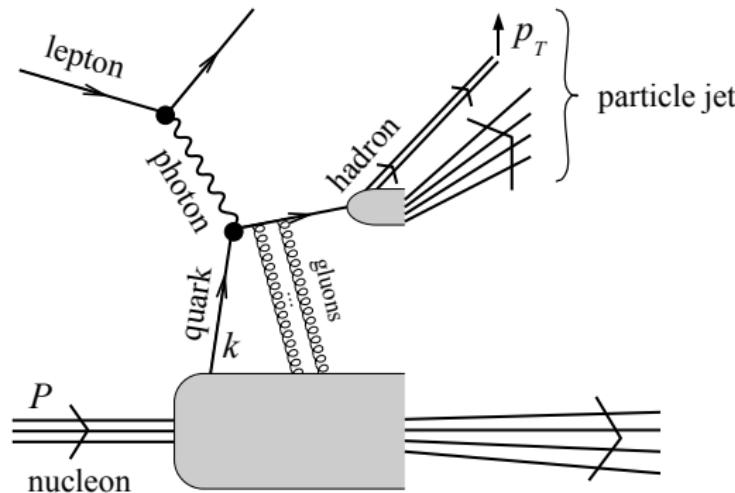
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- $x, \mathbf{k}_{\perp}$  (intrinsic transverse momentum)  
see talks by Burkardt, Jiang, Bacchetta, Hägler  
(Thursday morning)  $\Rightarrow$  TMD PDFs

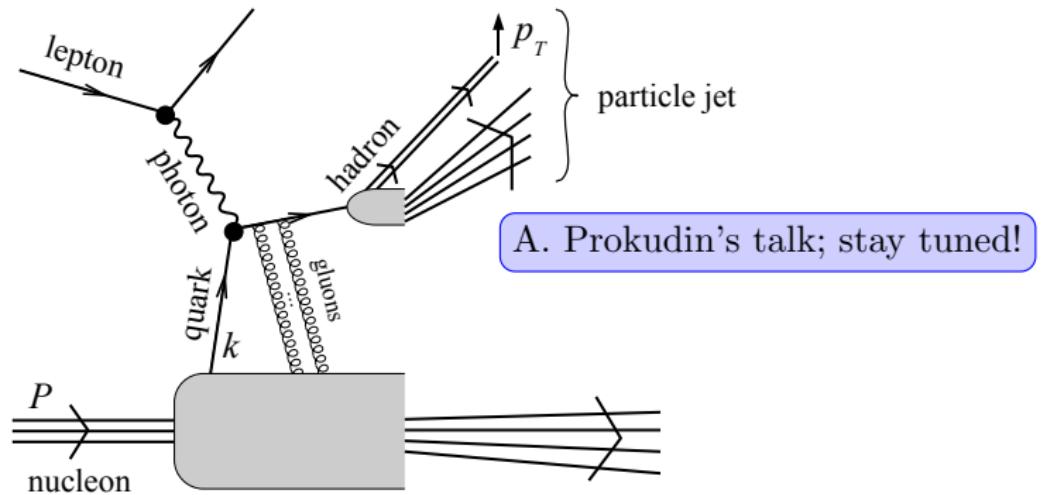
e.g., semi-inclusive DIS [COLLINS PLB 93], [BACCHETTA ET AL. JHEP 07]



experiments sensitive to TMD PDFs

COMPASS (CERN), HERMES (DESY), JLab, RHIC (BNL), Fermilab,  
also planned at J-PARC, FAIR (GSI), NICA (JINR), ..., EIC (BNL/JLab?)

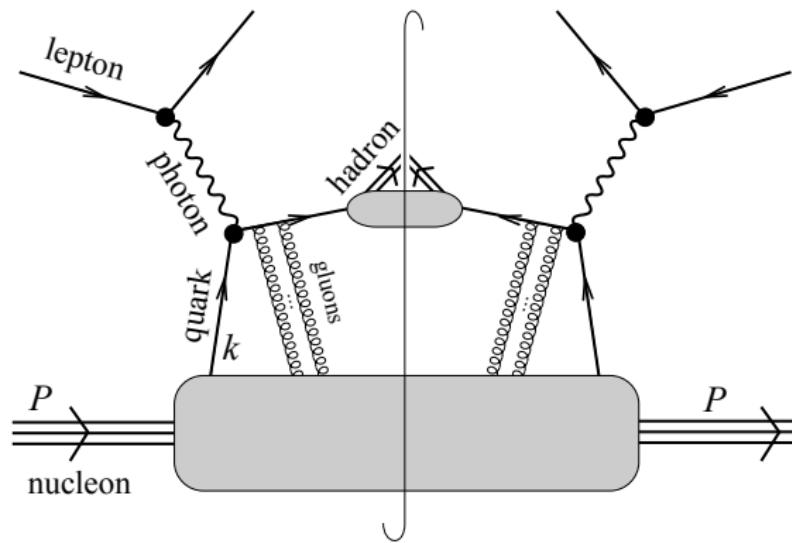
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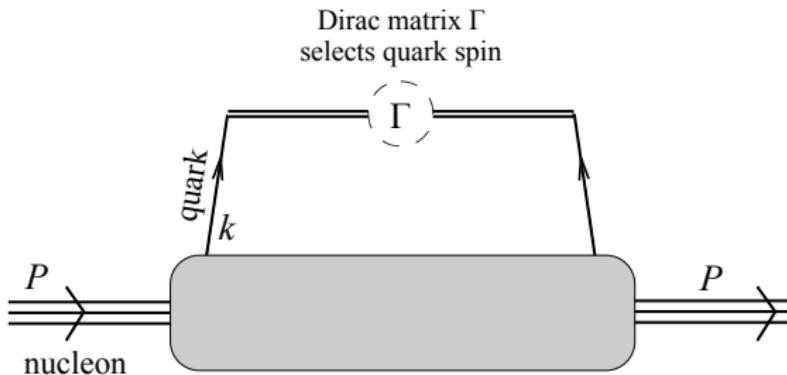
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$$\frac{d\sigma}{d^3 P_h d^3 P_{l'}} \propto \underbrace{H(Q^2, \dots)}_{\text{hard part}} \int d^2 \mathbf{k}_\perp \underbrace{f_1(x, \mathbf{k}_\perp, \dots)}_{\text{TMD PDF}} \underbrace{D_h(z, \mathbf{k}_\perp + \mathbf{q}_\perp, \dots)}_{\text{fragmentation f.}}$$

(no soft factor taken into account, see [JI, MA, YUAN PRD 71 (2005)])



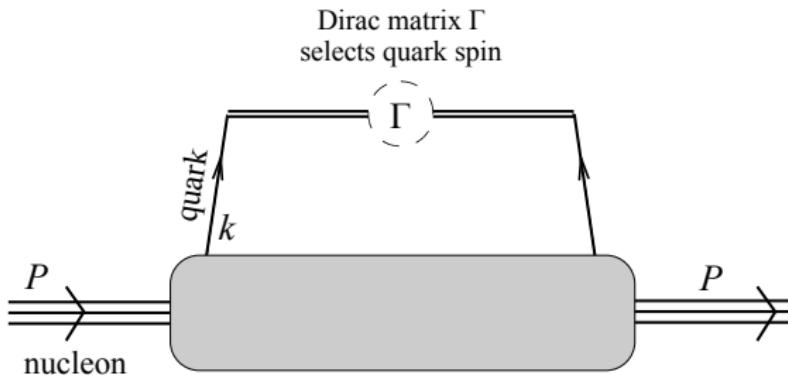
$$\Phi^{[\Gamma]}(k, P, S) \equiv “\langle P, S | \bar{q}(k) \Gamma q(k) | P, S \rangle”$$

lightcone coor.  $w^\pm = \frac{1}{\sqrt{2}}(w^0 \pm w^3)$ , so  $w = w^+ \hat{n}_+ + w^- \hat{n}_- + w_\perp$   
 proton flies along z-axis:  $P^+$  large,  $P_\perp = 0$

parametrization in terms of TMD PDFs, example

$$\int dk^- \Phi^{[\gamma^+]}\big(k, P, S\big)\Big|_{k^+=xP^+} = f_1(x, \mathbf{k}_\perp^2) - \frac{\epsilon_{ij} \mathbf{k}_i \mathbf{S}_j}{m_N} f_{1T}^\perp(x, \mathbf{k}_\perp)$$

[RALSTON, SOPER NPB 1979], [MULDERS, TANGERMAN NPB 1996], [GOEKE, METZ, SCHLEGEL PLB 2005]



$$\Phi^{[\Gamma]}(k, P, S) \equiv \frac{1}{2} \int \frac{d^4\ell}{(2\pi)^4} e^{-ik\cdot\ell} \langle P, S | \bar{q}(\ell) \Gamma \mathcal{U} q(0) | P, S \rangle$$

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# gauge link operator $\mathcal{U}$

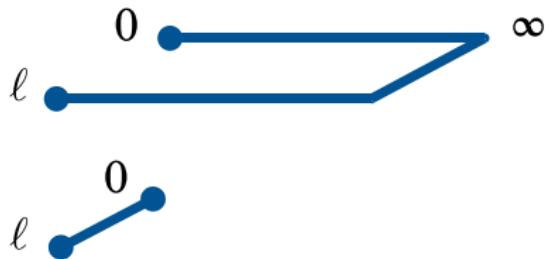
$\langle P | \bar{q}(\ell) \Gamma \mathcal{U} q(0) | P \rangle$  is gauge invariant.

continuum

$$\mathcal{U} \equiv \mathcal{P} \exp \left( -ig \int_0^\ell d\xi^\mu A_\mu(\xi) \right)$$

along path from 0 to  $\ell$

- factorization in SIDIS :  
path runs to infinity and back
- *simplification:*  
straight path (for first studies)



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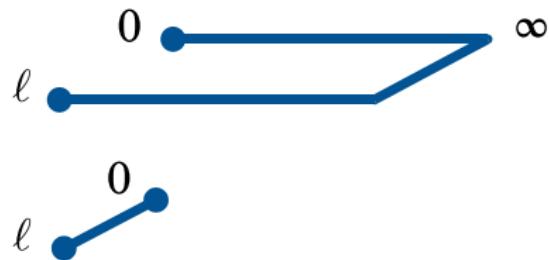
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→ L. Gamberg's talk  
at 3:20 pm

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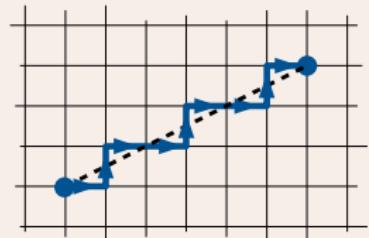
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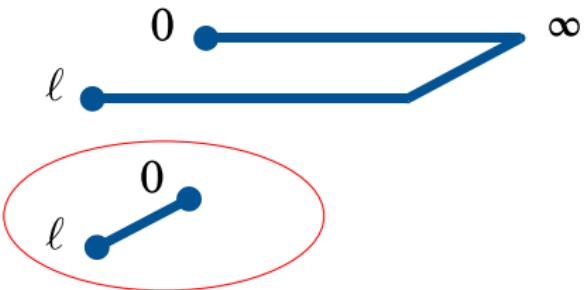
lattice



product of link variables

- factorization in SIDIS :  
path runs to infinity and back

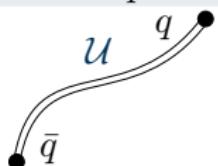
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continuum

[CRAIGIE, DORN NPB185,204 (1981)]

smooth path



$$[\bar{q} \mathcal{U} q]_{\text{ren}} = Z^{-1} \exp \left( -\delta \hat{m} \frac{l}{a} \right) [\bar{q} \mathcal{U} q]$$

$\delta \hat{m}$  : removes the length dependent renorm. factor

static quark potential

$$V_{\text{ren}}(r) = V(r) + 2 \delta \hat{m}/a$$

string [LÜSCHER, SYMANZIK, WEISZ (1980)]

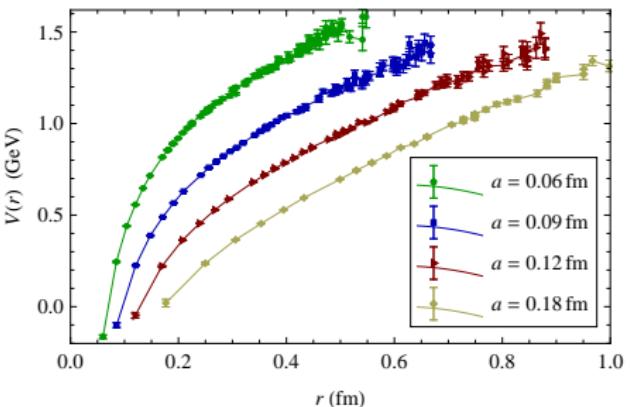
at large  $r$ :  $V_{\text{ren}}(r) \approx$ 

$$V_{\text{string}}(r) = \sigma r - \pi/12r + 0$$

method [CHENG PRD77,014511 (2008)]

determine  $\delta \hat{m}$  from

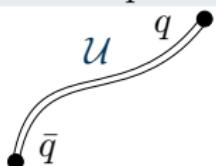
$$V_{\text{ren}}(0.7 \text{ fm}) \stackrel{!}{=} V_{\text{string}}(0.7 \text{ fm})$$



continuum

[CRAIGIE, DORN NPB185,204 (1981)]

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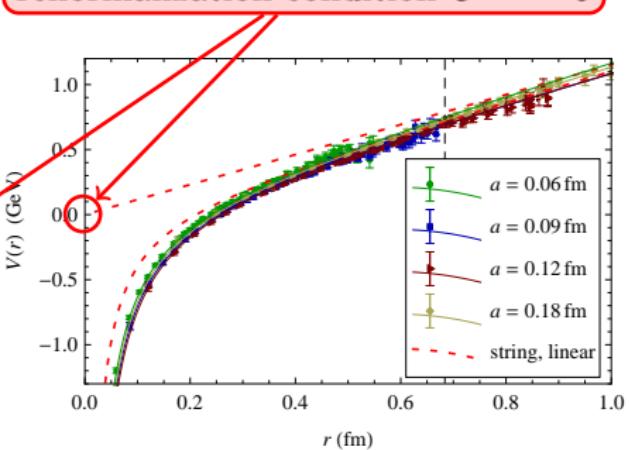
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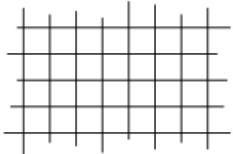
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$$V_{\text{ren}}(0.7 \text{ fm}) \stackrel{!}{=} V_{\text{string}}(0.7 \text{ fm})$$

renormalization condition  $C^{\text{ren}} = 0$ 

We employ the Chroma library [EDWARDS, JOO (2005)] to process



### MILC gauge configurations

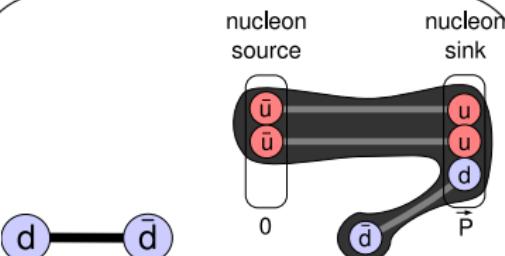
staggered Asqtad action,  
2+1 flavors,  $a \approx 0.124$  fm,  
 $m_\pi \approx 500, 610,$  and  $760$  MeV

[ORGINOS, TOUSSAINT PRD (1999)]

+ finer MILC lattices  
to test renormalization

[AUBIN ET AL. PRD (2004)]

[BAZAVOV ET AL. 0903.3598]

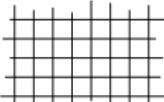
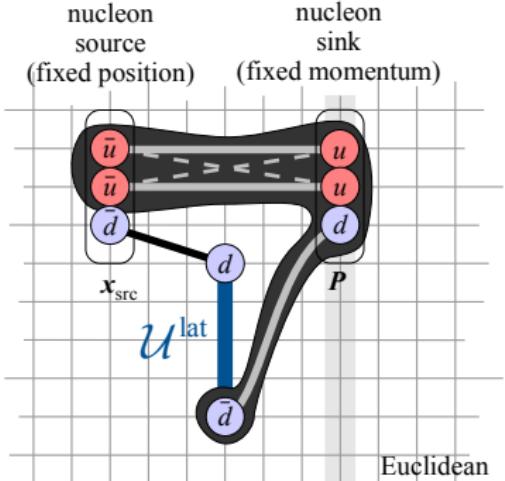
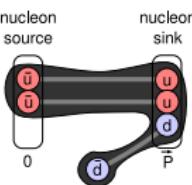


### LHPC propagators

domain wall valence fermions,  
 $m_\pi$  adjusted to staggered sea,  
nucleon momenta:

$$\mathbf{P} = 0 \text{ and } |\mathbf{P}| = 500 \text{ MeV}$$

e.g., [HÄGLER ET AL. PRD (2008)]

Ingredients	Output : 3-point correlator $C_{3\text{pt}}$
 gauge configs.	
 quark propagators	
 nucleon sequential propagators	<p>form ratio <math>C_{3\text{pt}}/C_{2\text{pt}}</math>, take plateau</p> $\Rightarrow \langle P, S   \bar{q}(\ell) \Gamma \mathcal{U} q(0)   P, S \rangle$

[We neglect “disconnected contributions” (absent for up minus down).]

$$\Phi^{[\Gamma]}(k, P, S) \equiv \frac{1}{2} \int \frac{d^4 \ell}{(2\pi)^4} e^{-ik \cdot \ell} \langle P, S | \bar{q}(\ell) \Gamma \mathcal{U} q(0) | P, S \rangle$$

isolation of Lorentz-invariant amplitudes

compare [MULDERS, TANGERMAN NPB (1996)]

$$\langle P, S | \bar{q}(\ell) \gamma_\mu \mathcal{U} q(0) | P, S \rangle = 4 \tilde{A}_2 P_\mu + 4i m_N^2 \tilde{A}_3 \ell_\mu$$

$$\begin{aligned} \langle P, S | \bar{q}(\ell) \gamma_\mu \gamma^5 \mathcal{U} q(0) | P, S \rangle &= -4 m_N \tilde{A}_6 S_\mu \\ &\quad -4i m_N \tilde{A}_7 P_\mu (\ell \cdot S) \\ &\quad +4 m_N^3 \tilde{A}_8 \ell_\mu (\ell \cdot S) \end{aligned}$$

$$\langle P, S | \bar{q}(\ell) \dots \mathcal{U} q(0) | P, S \rangle = \text{further structures (9 amplitudes in total)}$$

Transformation properties of the matrix element ( $\dagger, \mathcal{P}, \mathcal{T}$ ) limit number of allowed structures. No  $\mathcal{T}$ -odd structures (Sivers function, ...) with straight gauge link.

The amplitudes fulfill  $\tilde{A}_i(\ell^2, \ell \cdot P) = [\tilde{A}_i(\ell^2, -\ell \cdot P)]^*$ .

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$\Rightarrow f_1(x, \mathbf{k}_\perp^2)$

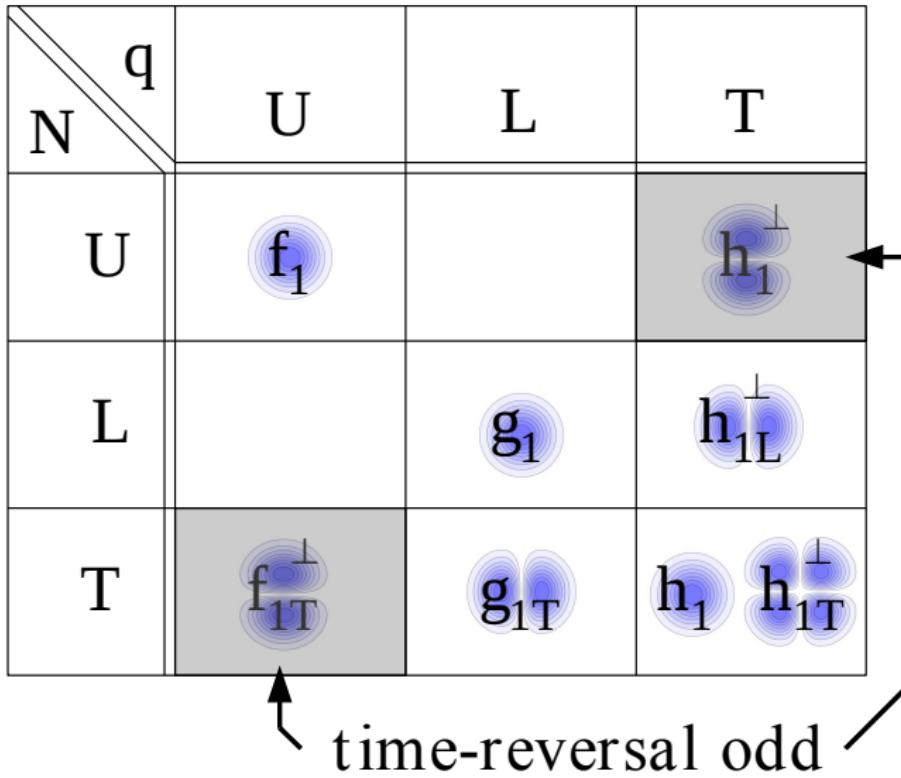
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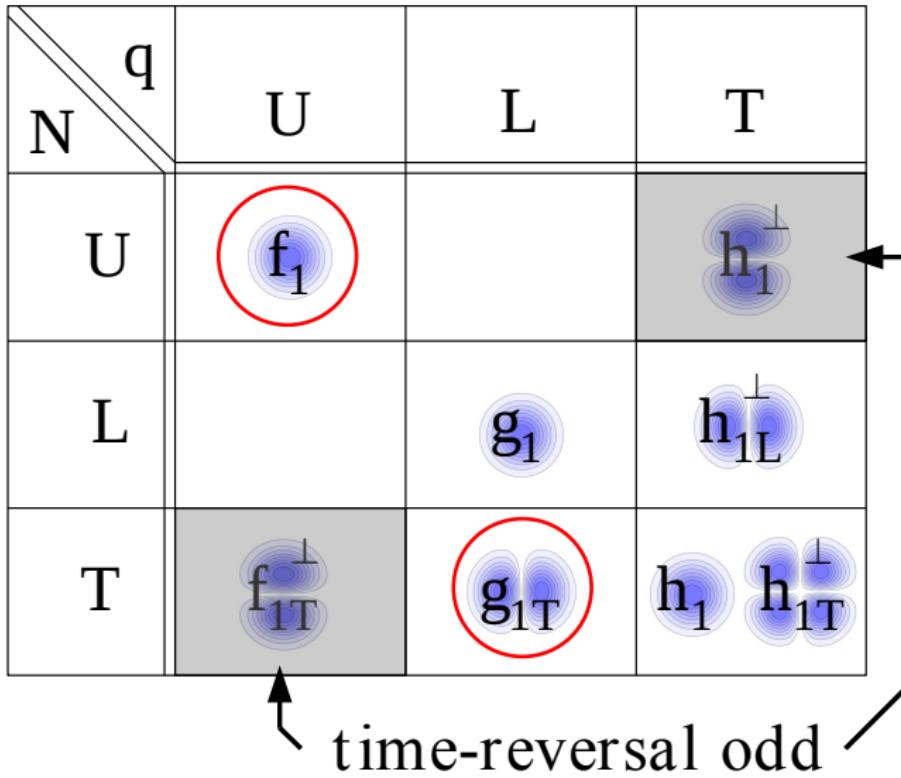
$\Rightarrow g_{1T}(x, \mathbf{k}_\perp^2)$

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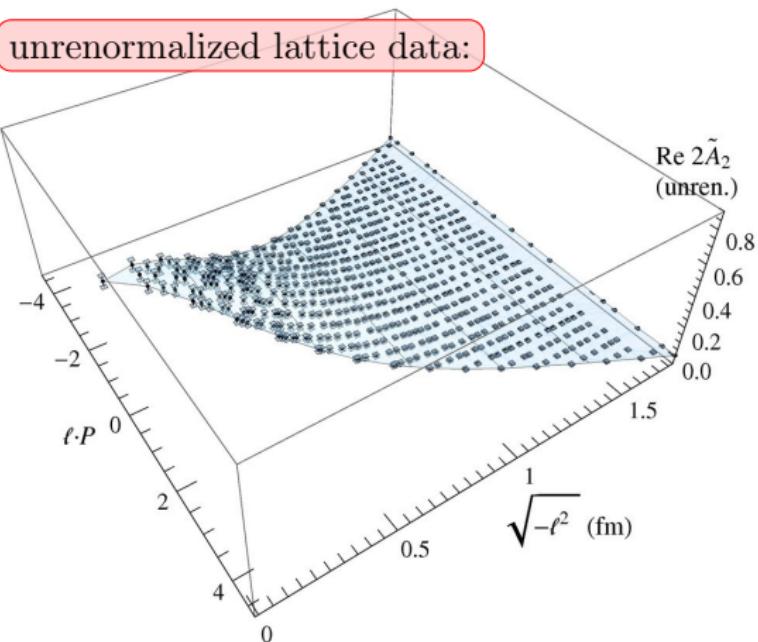


extract Lorentz-invariant amplitudes  $\tilde{A}_i(\ell^2, \ell \cdot P)$ , example :

$$\langle P, S | \bar{q}(\ell) \gamma_\mu \mathcal{U} q(0) | P, S \rangle = 4\tilde{A}_2 P_\mu + 4i m_N^2 \tilde{A}_3 \ell_\mu ,$$

$$f_1(x, k_\perp^2) = \int \frac{d(\ell \cdot P)}{2\pi} e^{ix(\ell \cdot P)} \int \frac{d^2 \ell_\perp}{(2\pi)^2} e^{-ik_\perp \cdot \ell_\perp} 2\tilde{A}_2(\ell^2, \ell \cdot P) \Big|_{\ell^+ = 0}$$

unrenormalized lattice data:



$$\ell^2 \xleftrightarrow{\text{FT}} k_\perp^2$$

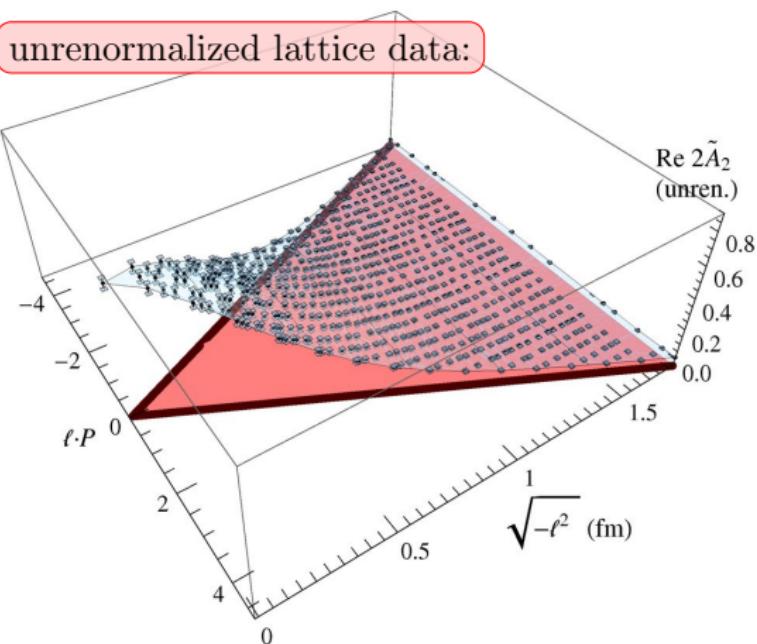
$$\ell \cdot P \xleftrightarrow{\text{FT}} x$$

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$$\ell \cdot P \xleftrightarrow{\text{FT}} x$$

Euclidean lattice

$$\ell^0 = \ell_4 = 0$$

$\Downarrow$

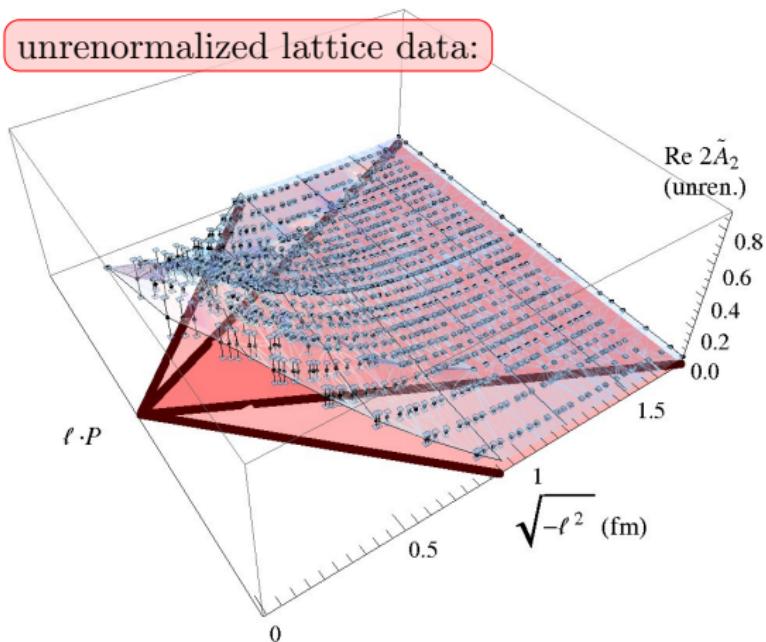
$$\ell^2 \leq 0, \\ |\ell \cdot P| \leq |\mathbf{P}| \sqrt{-\ell^2}$$

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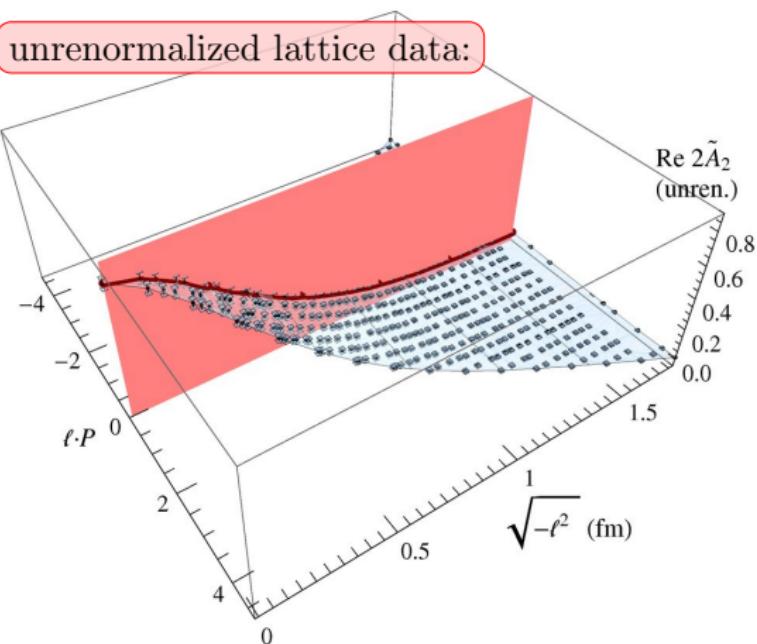
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$$\begin{array}{ccc} \ell^2 & \xleftrightarrow{\text{FT}} & k_\perp^2 \\ \ell \cdot P & \xleftrightarrow{\text{FT}} & x \end{array}$$

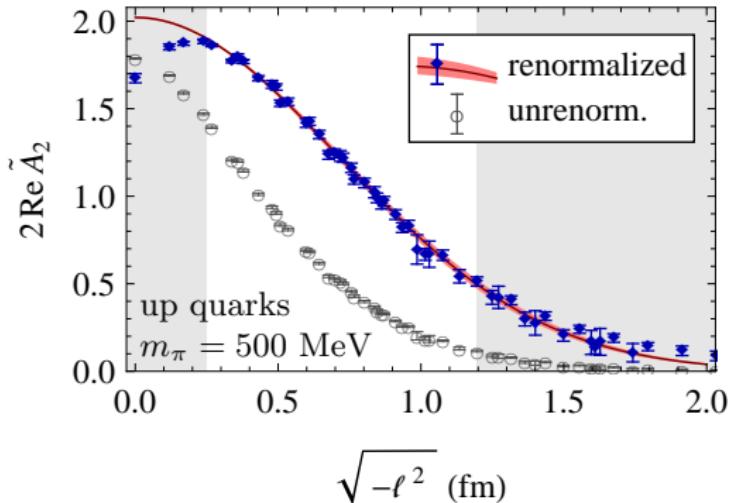
Euclidean lattice

$$\ell^0 = \ell_4 = 0$$

$\Downarrow$

$$\begin{aligned} \ell^2 &\leq 0, \\ |\ell \cdot P| &\leq |\mathbf{P}| \sqrt{-\ell^2} \end{aligned}$$

$$f_1^{(0_x)}(\mathbf{k}_\perp^2) \equiv \int_{-1}^1 dx f_1(x, \mathbf{k}_\perp^2) = \int \frac{d^2 \ell_\perp}{(2\pi)^2} e^{i \mathbf{k}_\perp \cdot \ell_\perp} 2 \tilde{A}_2(-\ell_\perp^2, 0)$$

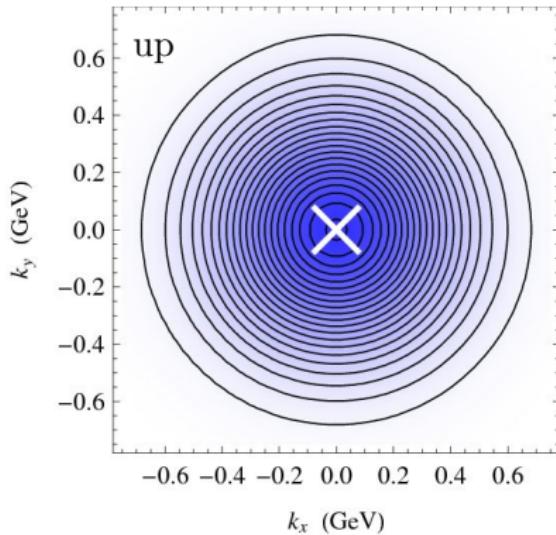


fit function  
 $C_1 \exp(-|\ell|^2/\sigma_1^2)$

Z-factor  
 $Z^{-1} C_1^{\text{up-down}} \stackrel{!}{=} 1$   
multiplicative  
renormalization based on  
quark counting

Density of unpolarized quarks (minus antiquarks)  
in an unpolarized nucleon as a function of transverse momentum  $\mathbf{k}_\perp$ :

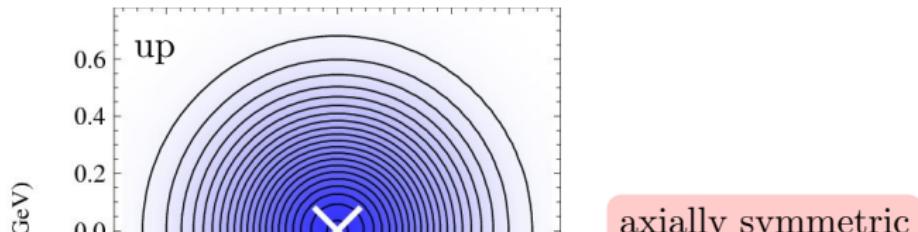
$$\rho_{UU}(\mathbf{k}_\perp) = \int_{-1}^1 dx f_1(x, \mathbf{k}_\perp^2)$$



axially symmetric

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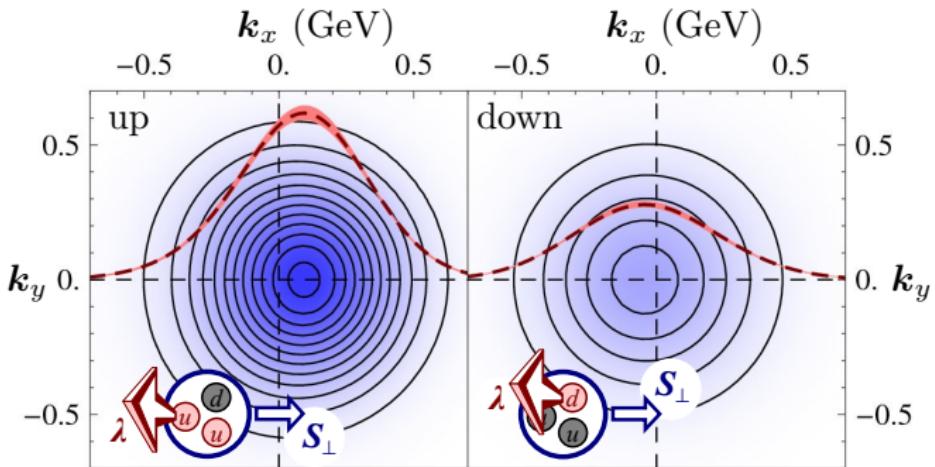
keep in mind



- correlator with straight Wilson line (“sW”)
- renormalized to string potential with  $C = 0$
- Gaussian fit ansatz  
 (“wrong” at large- $\mathbf{k}_\perp$  [DIEHL, arXiv:0811.0774])
- $m_\pi \approx 500$  MeV

Density of quarks with positive helicity,  $\lambda = 1$ ,  
in a transversely polarized nucleon,  $\mathbf{S}_\perp = (1, 0)$ :

$$\begin{aligned}\rho_{TL}(\mathbf{k}_\perp; \mathbf{S}_\perp, \lambda) &\equiv \frac{1}{2} \int dx \int dk^- \Phi^{[\gamma^+ \frac{1}{2}(\mathbb{1} + \gamma^5)]}(k, P, S_\perp) \\ &= \frac{1}{2} f_1^{(0_x)}(\mathbf{k}_\perp^2) + \frac{\lambda}{2} \frac{\mathbf{k}_\perp \cdot \mathbf{S}_\perp}{m_N} g_{1T}^{(0_x)}(\mathbf{k}_\perp^2)\end{aligned}$$



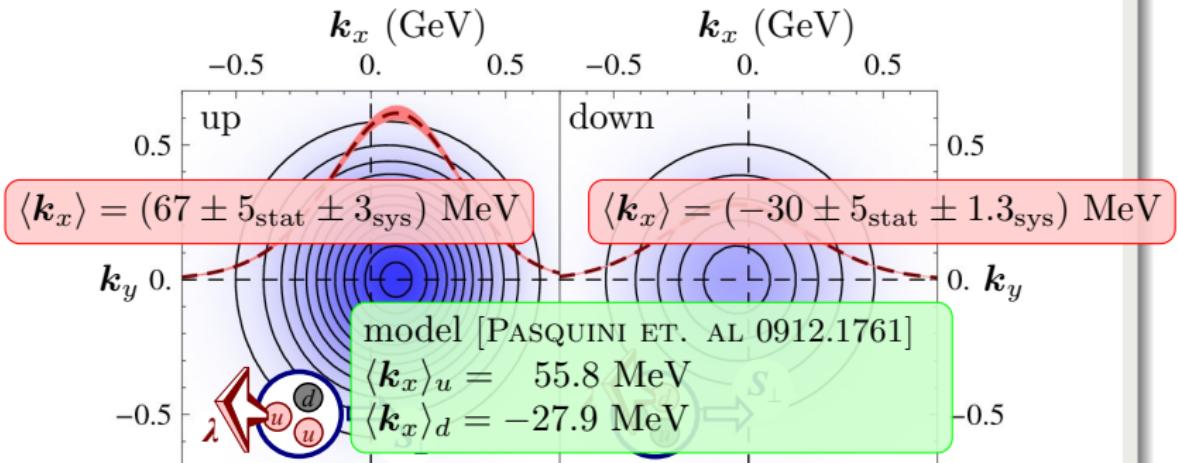
$(m_\pi \approx 500 \text{ MeV}, \text{ straight gauge link operator, } C^{\text{ren}} = 0, \text{ Gaussian fit})$

# a polarized $\mathbf{k}_\perp$ -dependent quark density

14

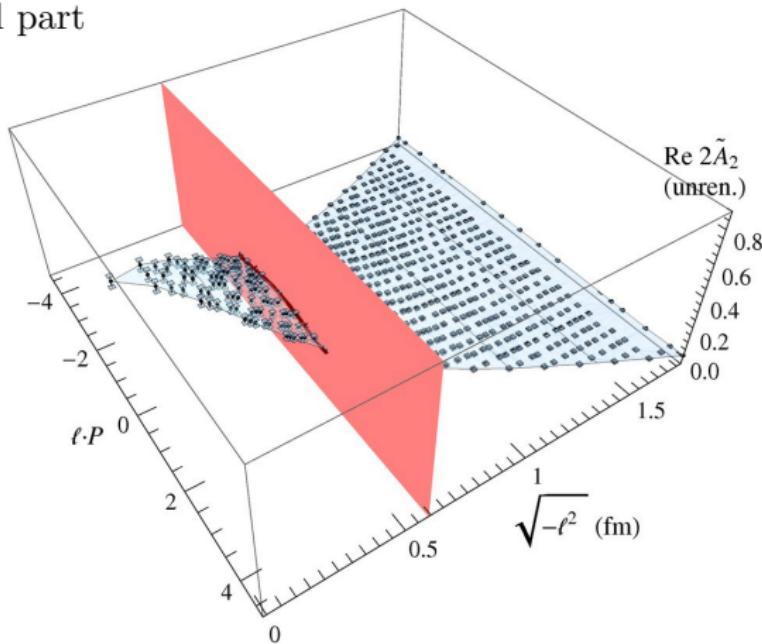
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$\left( \begin{array}{l} m_\pi \approx 500 \text{ MeV, straight gauge link operator,} \\ \text{renormalization condition } C^{\text{ren}} = 0, \text{ Gaussian fit} \end{array} \right)$

real part



$$\ell^2 \xleftrightarrow{\text{FT}} k_{\perp}^2$$

$$\ell \cdot P \xleftrightarrow{\text{FT}} x$$

factorization hypothesis

$$f_1(x, \mathbf{k}_\perp^2) \approx f_1(x) f_1^{(0_x)}(\mathbf{k}_\perp^2) / \mathcal{N}$$

as in phenomenological applications,  
e.g., Monte Carlo event generators

Then  $\tilde{A}_2$  factorizes, too:

$$\tilde{A}_2(\ell^2, \ell \cdot P) = \tilde{A}_2^{\text{norm}}(\ell \cdot P) \tilde{A}_2(\ell^2, 0).$$

To test this, we define

$$\tilde{A}_2^{\text{norm}}(\ell^2, \ell \cdot P) \equiv \frac{\tilde{A}_2(\ell^2, \ell \cdot P)}{\text{Re } \tilde{A}_2(\ell^2, 0)}$$

(needs no renormalization!)

If factorization holds,  $\tilde{A}_2^{\text{norm}}$  should be  $\ell^2$ -independent.

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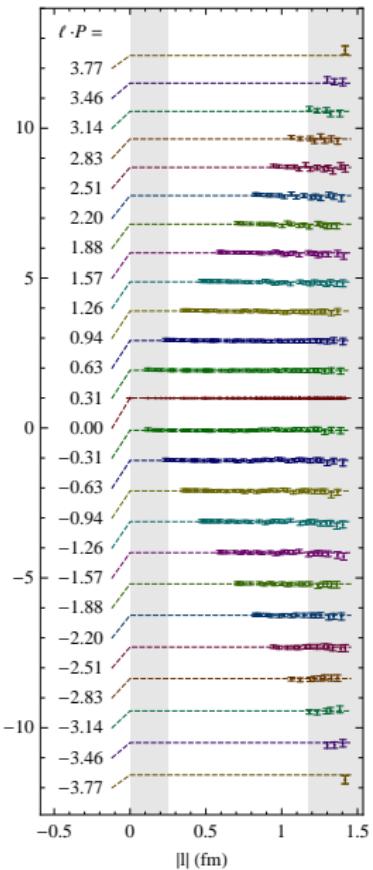
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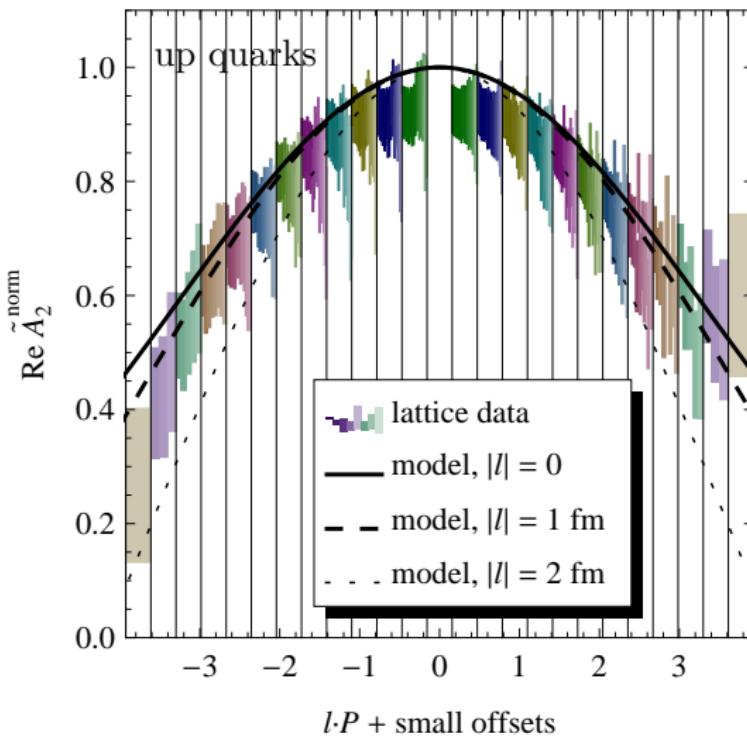
within statistics



All our data for  $\tilde{A}_2^{\text{norm}}(\ell^2, \ell \cdot P)$  at  $m_\pi \approx 600$  MeV

qualitative comparison to a scalar diquark model

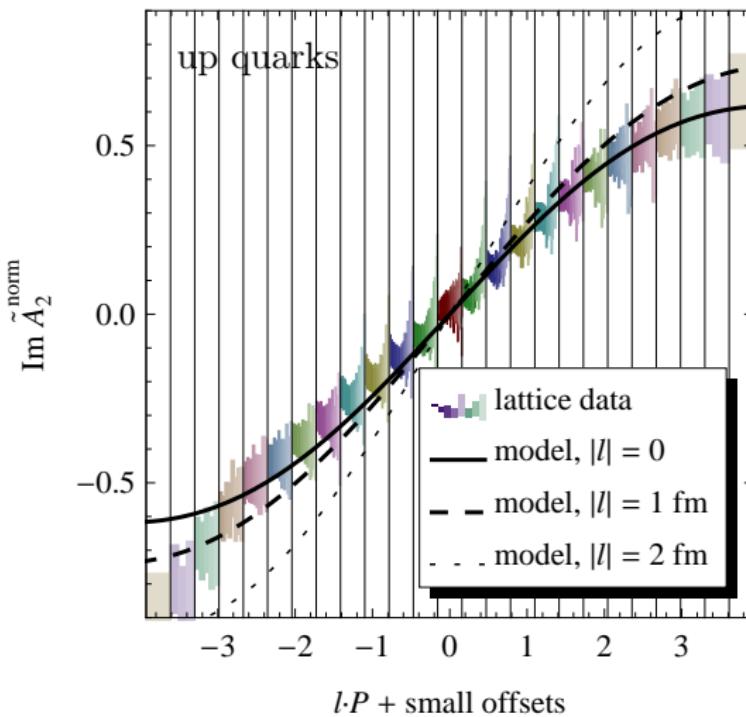
[BACCHETTA, CONTI, RADICI PRD (2008)] at  $\sqrt{-\ell^2} = 0, 1$  and  $2$  fm



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Summary:

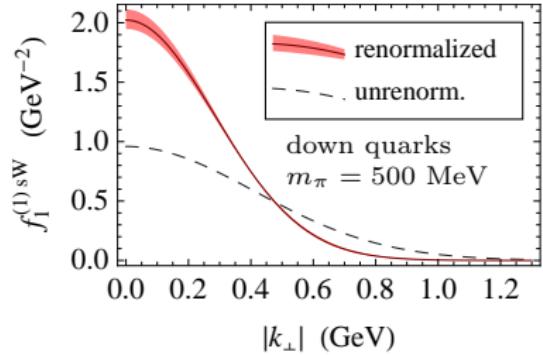
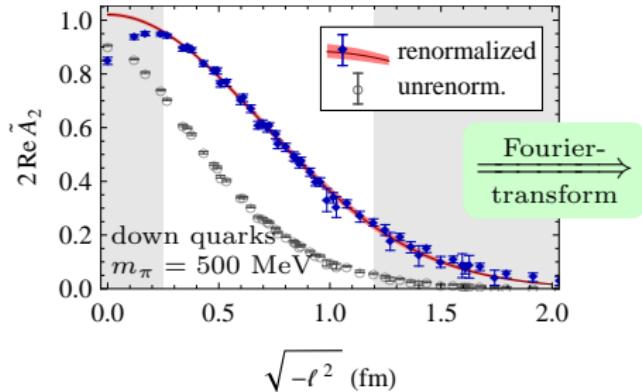
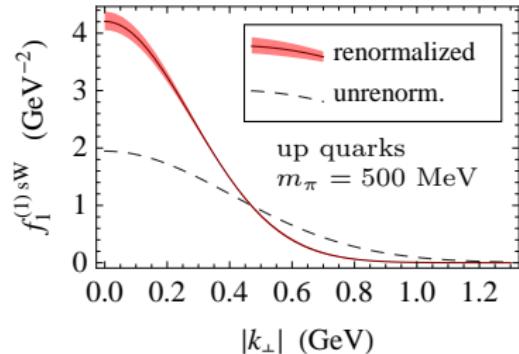
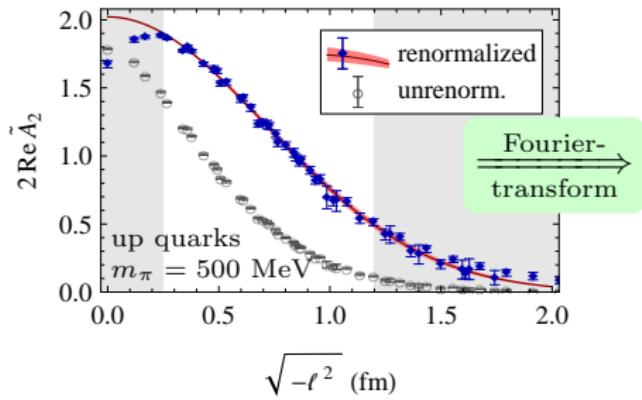
- We have explored ways to calculate intrinsic transverse momentum distributions in the nucleon with lattice QCD. We directly implement non-local operators on the lattice.
- First results are based on a simplified operator geometry (direct gauge link) and a Gaussian fit model, at  $m_\pi \approx 500$  MeV:
  - We calculate the first Mellin moment of leading twist TMD PDFs  $f_1^{(0)}(\mathbf{k}_\perp^2)$ ,  $g_{1T}^{(0)}(\mathbf{k}_\perp^2)$ ,  $h_{1L}^{\perp(1)}(\mathbf{k}_\perp^2)$  etc.
  - $\mathbf{k}_\perp$ -densities of longitudinally polarized quarks in a transversely polarized proton are deformed, due to non-vanishing  $g_{1T}^{(0)}$ .
  - So far, no statistically significant incompatibility with factorization  $f_1(x, \mathbf{k}_\perp^2) \approx f_1(x) f_1^{(0_x)}(\mathbf{k}_\perp^2)/\mathcal{N}$  detectable within the limited range of available lattice data  
→ higher statistics needed.

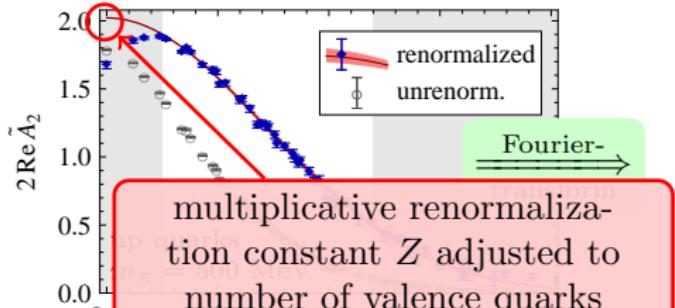
Outlook:

- Study of non-straight gauge links similar as in SIDIS.
- Beyond Gaussian fits:  
Matching to perturbative behavior at small  $\ell$ , i.e., large  $\mathbf{k}_\perp$ .

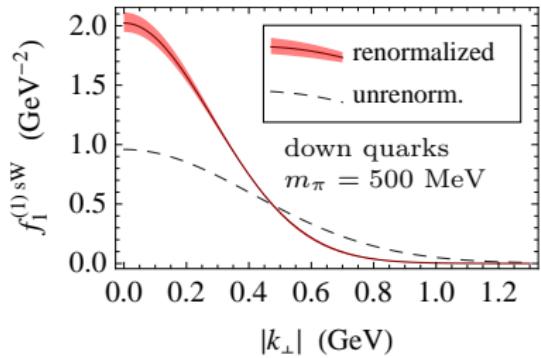
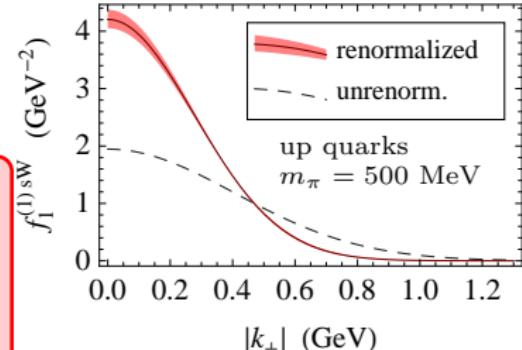
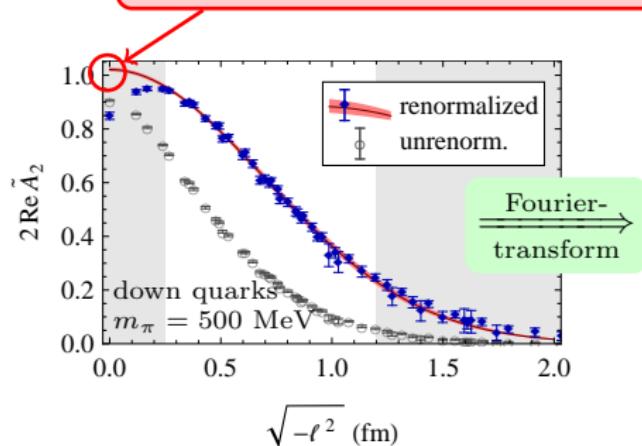


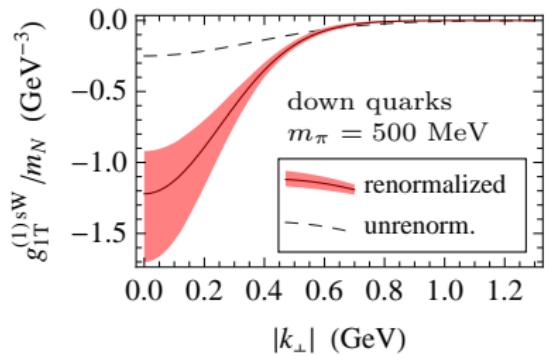
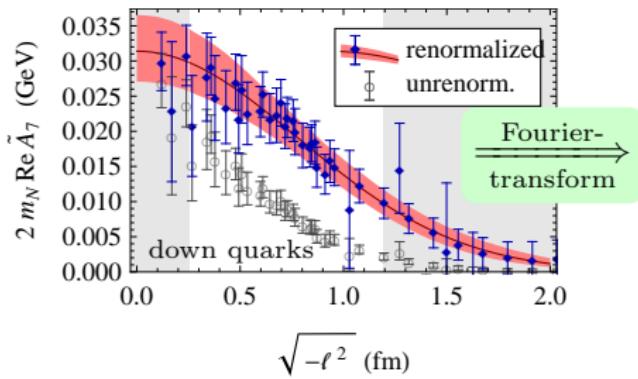
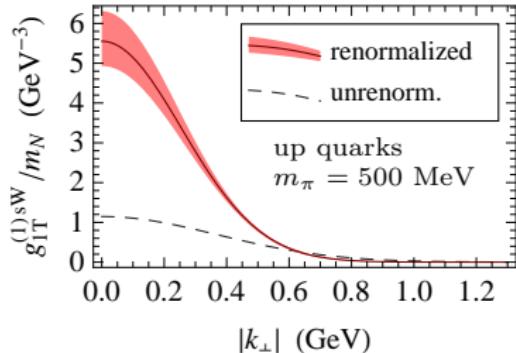
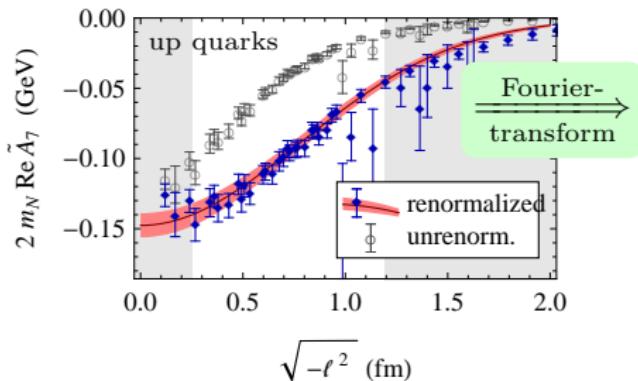
# Backup Slides

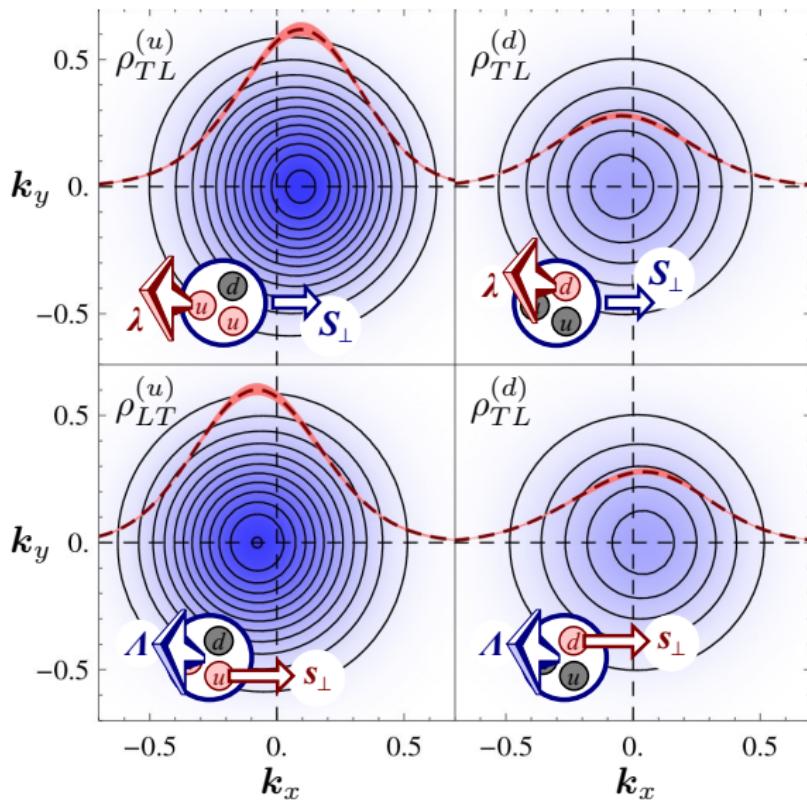




$$\int d^2 k_\perp f_1^{(0)}(k_\perp^2) = 2\tilde{A}_2(0,0), \\ \text{fixed in } u - d \text{ channel}$$







Dipole deformations

$$\rho_{TL} : \sim \lambda \mathbf{k}_\perp \cdot \mathbf{S}_\perp g_{1T}$$

$$\rho_{TL} : \sim \Lambda \mathbf{k}_\perp \cdot \mathbf{s}_\perp h_{1L}^\perp$$

The corresponding dipole structures  
 $\sim \lambda \mathbf{b}_\perp \cdot \mathbf{S}_\perp$ ,  
 $\sim \Lambda \mathbf{b}_\perp \cdot \mathbf{s}_\perp$   
for impact parameter densities (from GPDs)  
are ruled out by symmetries.

$$f^{(m_x, n_\perp)} \equiv \int_{-1}^1 dx x^m \int d^2 \mathbf{k}_\perp \left( \frac{\mathbf{k}_\perp^2}{2m_N^2} \right)^n f(x, \mathbf{k}_\perp^2)$$

Let us assume the amplitudes  $\tilde{A}_i$  are sufficiently regular at  $\ell^2 = 0$ .

$$\begin{aligned} \langle \mathbf{k}_\perp \rangle_{\rho_{TL}} &= \lambda \mathbf{S}_\perp m_N \frac{g_{1T}^{(0_x, 1_\perp)}}{f_1^{(0_x, 0_\perp)}} = \\ \lambda \mathbf{S}_\perp m_N \frac{\tilde{A}_7(0, 0)}{\tilde{A}_2(0, 0)} &\stackrel{?}{=} \lim_{\ell^2 \rightarrow 0} \lambda \mathbf{S}_\perp m_N \frac{\tilde{A}_7(\ell^2, 0)}{\tilde{A}_2(\ell^2, 0)} \end{aligned}$$

All self-energies from the gauge link cancel on the RHS  
 $(\Rightarrow$  no dependence on the renormalization condition).

Similar to weighted asymmetries from experiment ( $\rightarrow$  EIC):

$$A_{LT}^{\frac{Q_T}{m_N} \cos(\phi_h - \phi_S)} = 2 \frac{\langle \frac{Q_T}{m_N} \cos(\phi_h - \phi_S) \rangle_{UT}}{\langle 1 \rangle_{UU}} \propto \frac{\sum_q e_q^2 x g_{1T,q}^{(1_\perp)}(x) D_{1,q}(z)}{\sum_q e_q^2 x f_{1,q}(x) D_{1,q}(z)}$$

[BOER, MULDERS PRD 1998], [BACCHETTA ET AL. arXiv:1003.1328]

$$f_1^{(0_x)}(\mathbf{k}_\perp^2) = C_0 \exp(-\mathbf{k}_\perp^2/\mu_0^2)$$

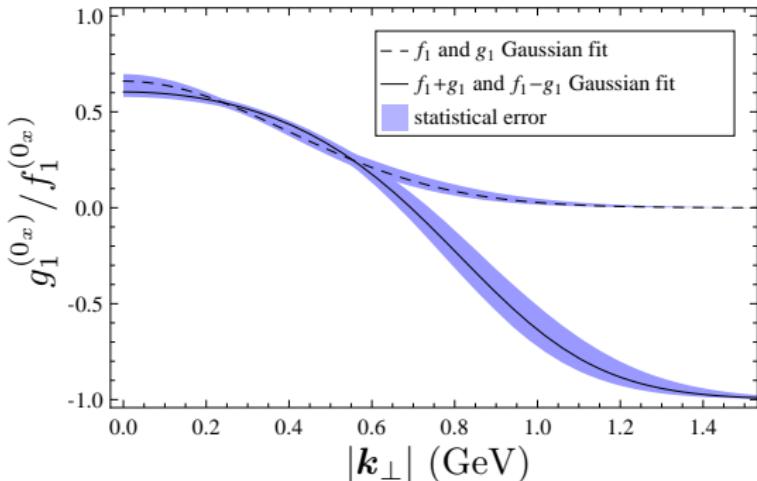
$$g_1^{(0_x)}(\mathbf{k}_\perp^2) = C_2 \exp(-\mathbf{k}_\perp^2/\mu_2^2)$$

vs.

$$\rho_{LL}^\pm(\mathbf{k}_\perp) \equiv \frac{1}{2} f_1^{(0_x)}(\mathbf{k}_\perp^2) \pm \frac{1}{2} g_1^{(0_x)}(\mathbf{k}_\perp^2)$$

$$\rho_{LL}^+(\mathbf{k}_\perp) = C_+ \exp(-\mathbf{k}_\perp^2/\mu_+^2)$$

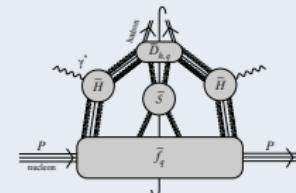
$$\rho_{LL}^-(\mathbf{k}_\perp) = C_- \exp(-\mathbf{k}_\perp^2/\mu_-^2)$$



⇒ Asymptotic behavior at large  $\mathbf{k}_\perp$  imposed by Gaussian ansatz;  
not a “lattice result”. Similar issues in analysis of experimental data.

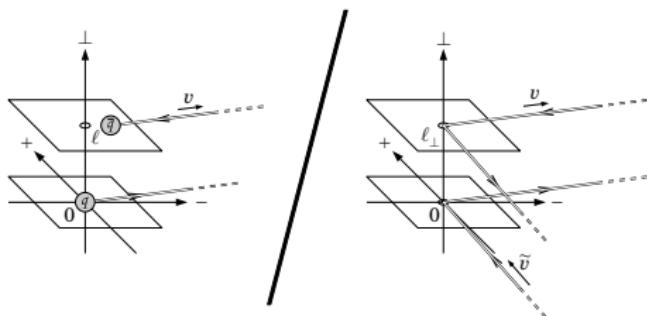
e.g., [JI, MA, YUAN PRD (2005)]:

$$W_{\text{unpol., LO}}^{\mu\nu} \propto H \times f_1 \otimes D_h \otimes \underbrace{\mathcal{S}}_{\text{soft factor}}$$



modified definition of TMD PDF correlator:

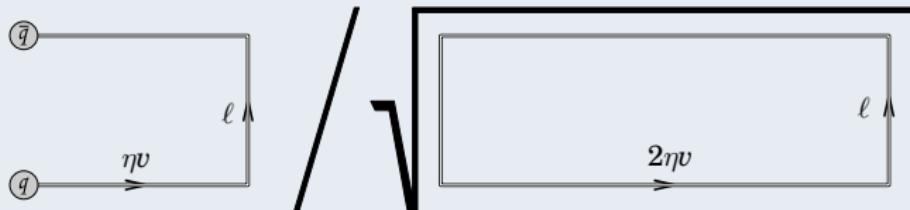
$$\Phi^{[\Gamma]}(k, P, S) \equiv \frac{1}{2} \int \frac{d^4 \ell}{(2\pi)^4} e^{-ik \cdot \ell} \frac{\langle P, S | \bar{q}(\ell) \Gamma \mathcal{U} q(0) | P, S \rangle}{\tilde{S}(\ell_\perp, \dots)}$$



- gauge links slightly off lightcone:  $v \neq \hat{n}_-$
- ⇒ evolution eqn. in  $\zeta \equiv (v \cdot P)^2/v^2$
- soft factor  $\tilde{S}$ : vacuum expectation value of gauge link structure

How to get rid of the gauge link self energy  $\exp(\delta m L)$ ?

Soft factor in TMD PDF correlator? Suggestion [COLLINS arXiv:0808.2665] :



Is this a meaningful definition of TMD PDFs?

prerequisite for quantitative lattice predictions

“To allow non-perturbative methods in QCD to be used to estimate parton densities, operator definitions of parton densities are needed that can be taken literally.” [COLLINS arXiv:0808.2665 (2008)]

$k_\perp$ -moments from ratios of amplitudes ...

... bridge the gap until we know more.

Example Sivers effect:  $\langle k_\perp \rangle_{\rho_{TU}}$  from  $\tilde{A}_{12}/\tilde{A}_2$ .

Self-energies cancel, no explicit subtraction factor needed.

ratio of correlators far away from nucleon source and sink

$$\frac{C_{3\text{pt}}(\tau; \Gamma, \ell, P)}{C_{2\text{pt}}(P)} \xrightarrow{t_{\text{src}} \ll \tau \ll t_{\text{sink}}} \begin{array}{l} \text{const. ("plateau value")}, \\ \downarrow \\ \text{access to } \langle P, S | \bar{q}(\ell) \Gamma \mathcal{U} q(0) | P, S \rangle \end{array}$$

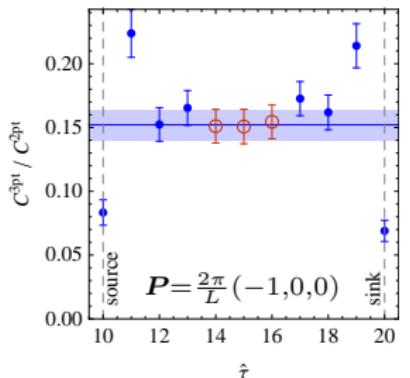
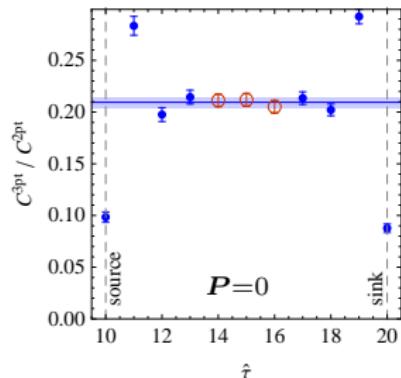
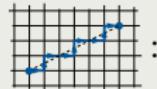
$\Gamma$	$\frac{1}{2} C_{3\text{pt}}^{\text{ren}}(\tau; \Gamma, \ell, P) / C_{2\text{pt}}(P)$ (LHPC projectors)
$\mathbb{1}$	$\frac{m_N}{E(P)} \tilde{A}_1$
$-\gamma_4 \gamma_5$	$i m_N \tilde{A}_7 \ell_z$
$\gamma_4$	$\tilde{A}_2$
$\frac{1}{2}[\gamma_2, \gamma_4]$	$\frac{1}{E(P)} \tilde{A}_9 P_x + \frac{i m_N^2}{E(P)} \tilde{A}_{10} \ell_x + \frac{m_N^2}{E(P)} \tilde{A}_{11} (\ell_z)^2 P_x$
...	...

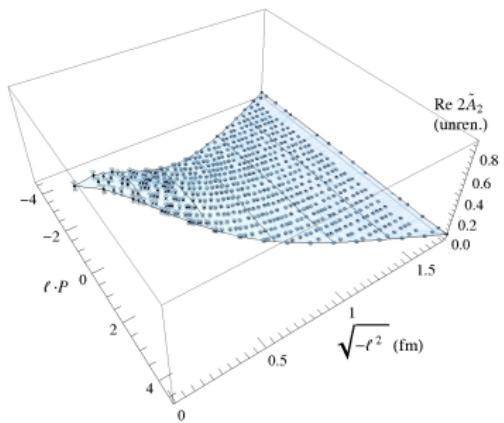
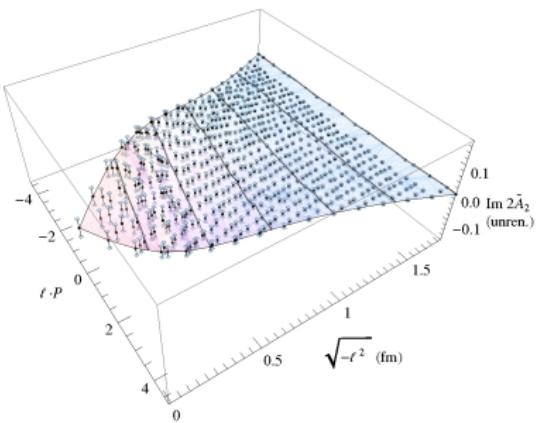
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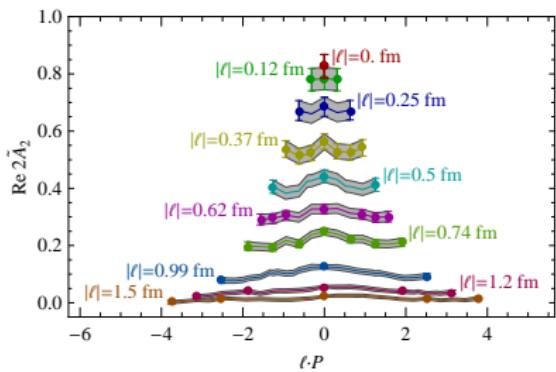
example plateau plots at  $m_\pi \approx 600$  MeV

for  $\Gamma = \gamma_4$  ( $\Rightarrow \tilde{A}_2$ ), with HYP smeared gauge link  $\mathcal{U} =$  :

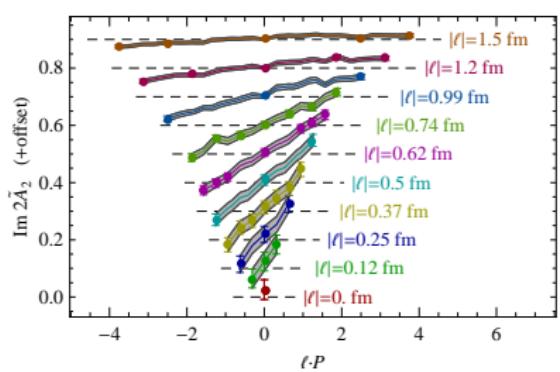


2 Re  $\tilde{A}_2(\ell^2, \ell \cdot P)$ 2 Im  $\tilde{A}_2(\ell^2, \ell \cdot P)$ 

2 Re  $\tilde{A}_2(\ell^2, \ell \cdot P)$

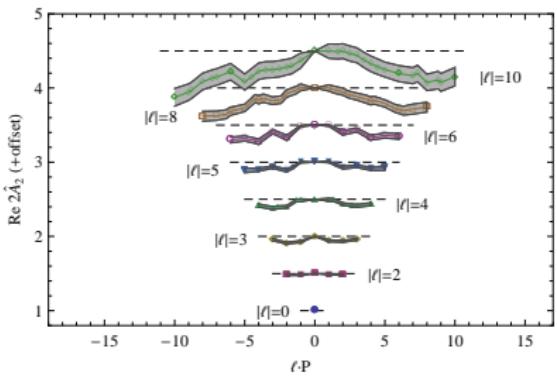
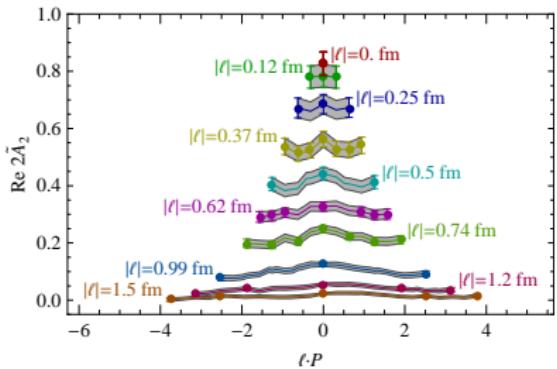


2 Im  $\tilde{A}_2(\ell^2, \ell \cdot P)$



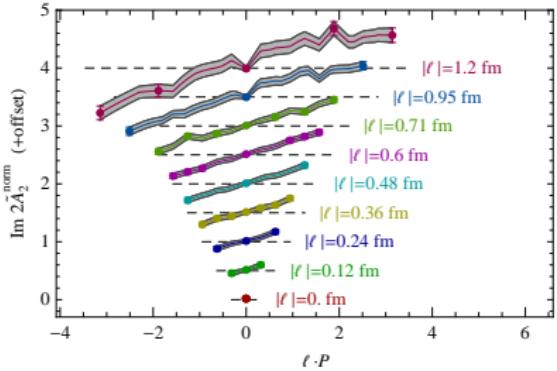
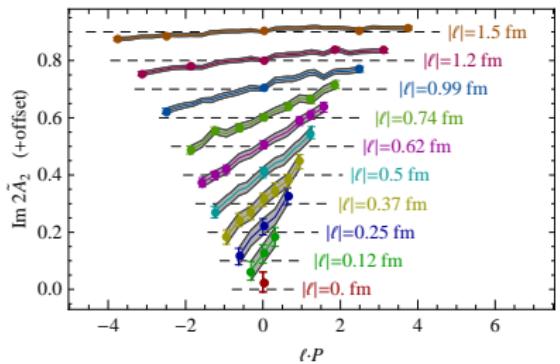
$$2 \operatorname{Re} \tilde{A}_2(\ell^2, \ell \cdot P)$$

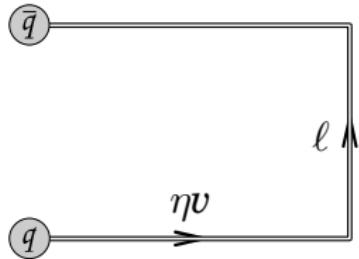
$$\operatorname{Re} \tilde{A}_2^{\text{norm}} = \frac{\operatorname{Re} \tilde{A}_2(\ell^2, \ell \cdot P)}{\operatorname{Re} \tilde{A}_2(\ell^2, 0)}$$



$$2 \operatorname{Im} \tilde{A}_2(\ell^2, \ell \cdot P)$$

$$\operatorname{Im} \tilde{A}_2^{\text{norm}} = \frac{\operatorname{Im} \tilde{A}_2(\ell^2, \ell \cdot P)}{\operatorname{Re} \tilde{A}_2(\ell^2, 0)}$$





### 32 Lorentz-invariant amplitudes [GOEKE,METZ,SCHLEGEL PLB618,90 (2005)]

$$A_i\left(k^2, k \cdot P, \frac{v \cdot k}{|v \cdot P|}, \frac{v^2}{|v \cdot P|^2}, \frac{v \cdot P}{|v \cdot P|}\right) = A_i\left(k^2, k \cdot P, \underbrace{\frac{v \cdot k}{|v \cdot P|}}_{\approx x}, \zeta^{-1}, \text{sgn}(v \cdot P)\right)$$

Links approaching light cone:  $v \rightarrow \hat{n}_- \Rightarrow \zeta \rightarrow \infty$ . For large  $\zeta$ , the evolution with  $\zeta$  is known [COLLINS,SOPER NPB194,445 (1981)].

$$\left. \begin{array}{c} (v^0, v^1, v^2, v^3) \\ \text{future pointing } v \\ \text{TMD PDFs for SIDIS} \end{array} \right\} \xrightarrow{\mathcal{T}} \left\{ \begin{array}{c} (-v^0, v^1, v^2, v^3) \\ \text{past pointing } v \\ \text{TMD PDFs for Drell-Yan} \end{array} \right.$$

The transformation property of the matrix elements under time reversal provides relations:

Example of a  $\mathcal{T}$ -even amplitude:

$$A_2\left(k^2, k \cdot P, \frac{v \cdot k}{v \cdot P}, \zeta^{-1}, 1\right) = A_2\left(k^2, k \cdot P, \frac{v \cdot k}{v \cdot P}, \zeta^{-1}, -1\right)$$

$\Downarrow$

$$f_1^{(\text{SIDIS})}(x, \mathbf{k}_\perp; \zeta, \dots) = f_1^{(\text{Drell-Yan})}(x, \mathbf{k}_\perp; \zeta, \dots)$$

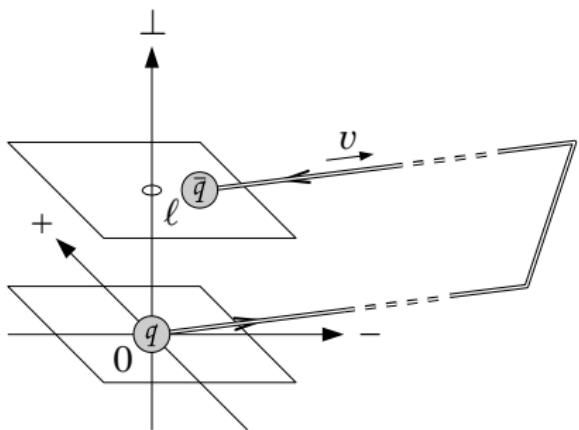
Example of a  $\mathcal{T}$ -odd amplitude: ( $\rightarrow$  Sivers function  $f_{1T}^\perp$ )

$$A_{12}\left(k^2, k \cdot P, \frac{v \cdot k}{v \cdot P}, \zeta^{-1}, 1\right) = -A_{12}\left(k^2, k \cdot P, \frac{v \cdot k}{v \cdot P}, \zeta^{-1}, -1\right)$$

$\Downarrow$

$$f_{1T}^{\perp(\text{SIDIS})}(x, \mathbf{k}_\perp; \zeta, \dots) = -f_{1T}^{\perp(\text{Drell-Yan})}(x, \mathbf{k}_\perp; \zeta, \dots)$$

- ... appear in factorized SIDIS / Drell-Yan process
- are responsible for “time-reversal-odd” TMD PDFs, such as  $f_{1T}^\perp$  (Sivers-function)

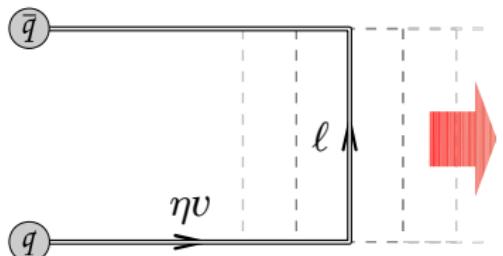


- gauge link = effective representation of struck quark (“final state interaction”)

- $\Rightarrow$  (almost lightlike)

$$\zeta \equiv \frac{(v \cdot P)^2}{v^2} \rightarrow \pm\infty$$

- keep  $\zeta$  finite to avoid “rapidity divergences”
- evolution equation in  $\zeta$  [COLLINS, SOPER NPB (1981)]

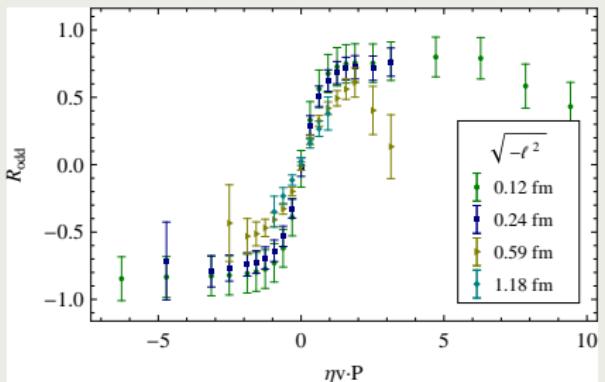


- $v$  spatial  $\Rightarrow |\zeta| = \frac{(v \cdot P)^2}{|v|^2} \leq |\mathbf{P}_{\text{lat.}}|^2$
- look for plateaus at large  $|\eta|$
- now 32 amplitudes

[GOEKE, METZ, SCHLEGEL PLB (2005)]

$$\tilde{a}_i(\ell^2, \ell \cdot P, v \cdot P; \eta, \zeta), \tilde{b}_i(\dots)$$

Test calculation: a time reversal odd ratio of amplitudes



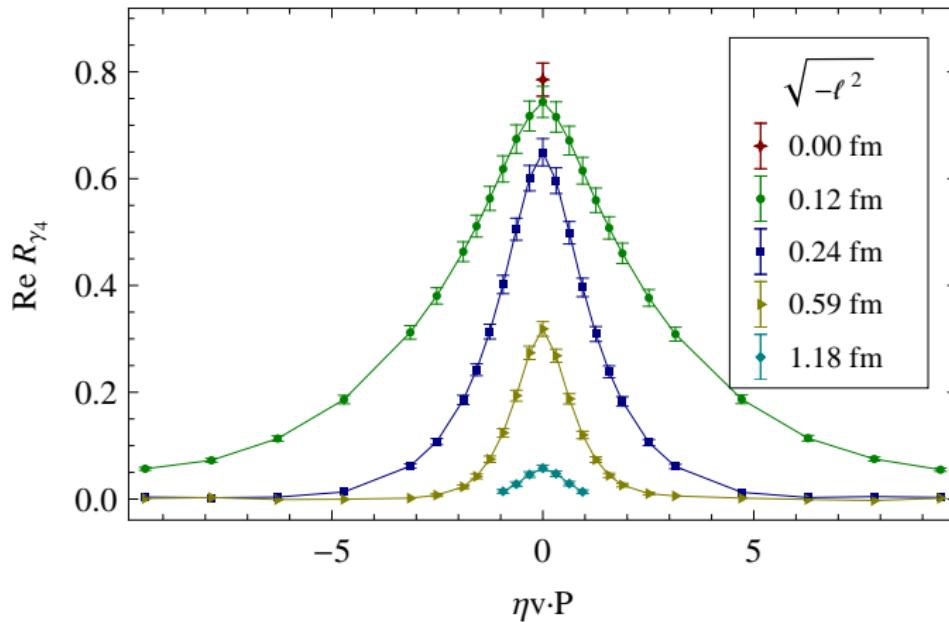
$$R_{\text{odd}} = -\frac{\tilde{a}_{12} - (\eta \frac{m_N^2 v_1}{P_1}) \tilde{b}_8}{\tilde{a}_2}$$

Plateaus visible at large  $|\eta|$ .

“Time-reversal odd”  $\leftrightarrow$   
odd in  $\eta v \cdot P$ .

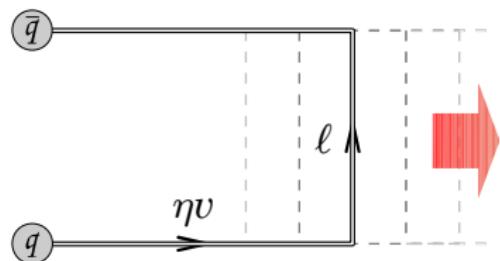
Part of the effect comes from  
the Sivers function  $f_{1T}^\perp$  !

$$\tilde{A}_2 \left( \ell^2, \ell \cdot P, \frac{v \cdot \ell}{|v \cdot P|}, \zeta^{-1}, \text{sgn}(v \cdot P) \right) \equiv \lim_{\eta \rightarrow \infty} \tilde{a}_2(\ell^2, \ell \cdot P, \eta v \cdot \ell, -\eta^2, \eta v \cdot P)$$



But  $\tilde{a}_2 = \text{Re } R_{\gamma_4}$  always vanishes for large  $\eta$ !

Reason: power divergence suppresses  $\tilde{a}_2 \sim \exp(-\delta m \eta)$ .



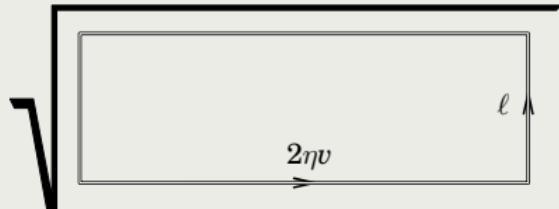
- $v$  spatial  $\Rightarrow |\zeta| = \frac{(v \cdot P)^2}{|v|^2} \leq |\mathbf{P}_{\text{lat.}}|^2$
- look for plateaus at large  $|\eta|$
- now 32 amplitudes  
 $\tilde{a}_i(\ell^2, \ell \cdot P, v \cdot P; \eta, \zeta)$

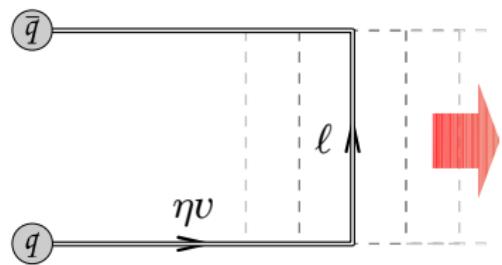
**Problem:** need to subtract gauge link self-energy ( $\rightarrow \eta$ -independence)

idea #1: modify definition of TMD PDFs [COLLINS PoS LC (2008)]

$$\Phi^{[\Gamma]}(k, P, S) \equiv \frac{1}{2} \int \frac{d^4 \ell}{(2\pi)^4} e^{-ik \cdot \ell} \frac{\langle P, S | \bar{q}(\ell) \Gamma \mathcal{U} q(0) | P, S \rangle}{\tilde{S}(\ell_\perp, \dots)}$$

with  $\tilde{S}$  obtained from a vacuum expectation value of gauge links, e.g.,





- $v$  spatial  $\Rightarrow |\zeta| = \frac{(v \cdot P)^2}{|v|^2} \leq |\mathbf{P}_{\text{lat.}}|^2$
- look for plateaus at large  $|\eta|$
- now 32 amplitudes  
 $\tilde{a}_i(\ell^2, \ell \cdot P, v \cdot P; \eta, \zeta)$

**Problem:** need to subtract gauge link self-energy ( $\rightarrow \eta$ -independence)

idea #2: ratios of amplitudes  $\rightarrow$  certain  $\mathbf{k}_\perp$ -moments

e.g., formally,

$$\langle \mathbf{k}_y \rangle_{TU} = -2m_N \mathbf{S}_x \lim_{\eta \rightarrow \infty} \frac{\tilde{a}_{12}(0, 0, 0; \eta, \zeta) + \dots}{\tilde{a}_2(0, 0, 0; \eta, \zeta)} \propto \frac{\int dx \int d^2 \mathbf{k}_\perp \mathbf{k}_\perp^2 f_{1T}^\perp}{\int dx \int d^2 \mathbf{k}_\perp f_1}$$

**Sivers function** causes average transverse quark momentum in  $y$ -direction in a transversely polarized nucleon (spin in  $x$ -direction).

$$\langle \mathbf{k}_y \rangle_{TU} \underset{\eta \text{ large}}{\approx} -2m_N \mathbf{S}_x \frac{\tilde{a}_{12}(\ell_{\min}^2, 0, 0; \eta, \zeta) + \dots}{\tilde{a}_2(\ell_{\min}^2, 0, 0; \eta, \zeta)} \quad \text{Self-energy cancels!}$$